PHYC 581: High Energy Astrophysics
Fall 2018

Lecture 4: 08/29/2018

Some Relativistic Effects:

High energy astrophysics deals with the behavior of matter moving at close to light speed, or trapped within the deep potential well of sources with strong gravitational field. In many cases, both of these conditions are present in the phenomena producing the energetic photons we detect. Hence, it is essential to include the effects of special and general relativity into description of the physics underlying the interactions between particles and radiation.

For a detailed introduction to special and general relativity one can refer to one of the many excellent books on these subjects. Here we briefly discuss some of the relativistic effects that are important in high energy astrophysics.

Relativistic Doppler Shift:

The four-momentum of a particle with mass m and velocity $\vec{v}$ is defined as follows:

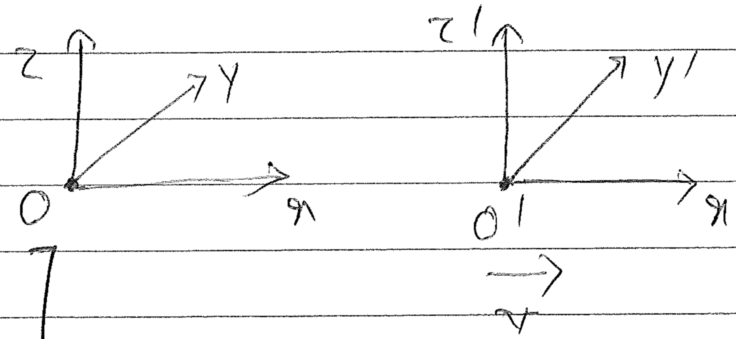
$$p^\mu = \left(\frac{E}{c}, \vec{p}\right) \qquad \vec{p} = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}} \qquad E = \sqrt{(pc)^2 + (mc^2)^2}$$

$$p \equiv |\vec{p}|$$

The four-momentum is an important example of a four-vector in special relativity. Under Lorentz transformations, it changes in the same way as spacetime coordinates x^μ.

For example, between two inertial frames with a relative velocity in the x direction, we have:

$$p'^\mu = a^\mu{}_\nu \, p^\nu$$



$$a^\mu{}_\nu \equiv \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \beta \equiv \frac{v}{c}, \quad \gamma \equiv \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

For a massless particle, like the photon, we have:

$E = pc$

The energy of a photon is related to its frequency ν according to $E = h\nu$. Now consider a source that emits photons of frequency ν as measured by an observer in the rest frame of that source. A detector moving with velocity $\vec{v}$ in the x direction, relative to the source, detects the emitted photons at an angle θ from the x axis. The frequency of detected photons is ν'.

Identifying the rest frames of the source and detector with O and O' on the previous page, we have:

$$\frac{E}{c} = \frac{\frac{E'}{c} + \frac{v}{c} p'_x}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad E = h\nu,\ E' = h\nu',\ p'_x = p'\cos\theta',\ p' = \frac{E'}{c}$$

Thus:

$$\frac{h\nu}{c} = \frac{\frac{h\nu'}{c} + \frac{v}{c}\frac{h\nu'}{c}\cos\theta'}{\sqrt{1 - \frac{v^2}{c^2}}} \Rightarrow \nu' = \frac{\sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{v}{c}\cos\theta'}\,\nu$$

Three special cases can arise that are of interest:

- $\theta' = 0$. This corresponds to the source moving away from the detector. In this case, we have:

$$\cos\theta' = 1 \Rightarrow \nu' = \frac{\sqrt{1-\frac{v^2}{c^2}}}{1+\frac{v}{c}}\nu \Rightarrow \nu' = \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}\,\nu \Rightarrow \nu' < \nu$$

This is the formula for the relativistic "Doppler redshift".

- $\theta' = \pi$. This corresponds to the source moving toward the detector. In this case, we have:

$$\cos\theta' = -1 \Rightarrow \nu' = \frac{\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{v}{c}}\nu \Rightarrow \nu' = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}\,\nu \Rightarrow \nu' > \nu$$

This is the formula for the relativistic "Doppler blueshift".

- $\theta' = \frac{\pi}{2}$. This corresponds to the source moving in the transverse direction. In this case, we have:

$$\cos\theta' = 0 \Rightarrow \nu' = \sqrt{1-\frac{v^2}{c^2}}\,\nu \Rightarrow \nu' < \nu$$

This is the formula for the "transverse Doppler effect".

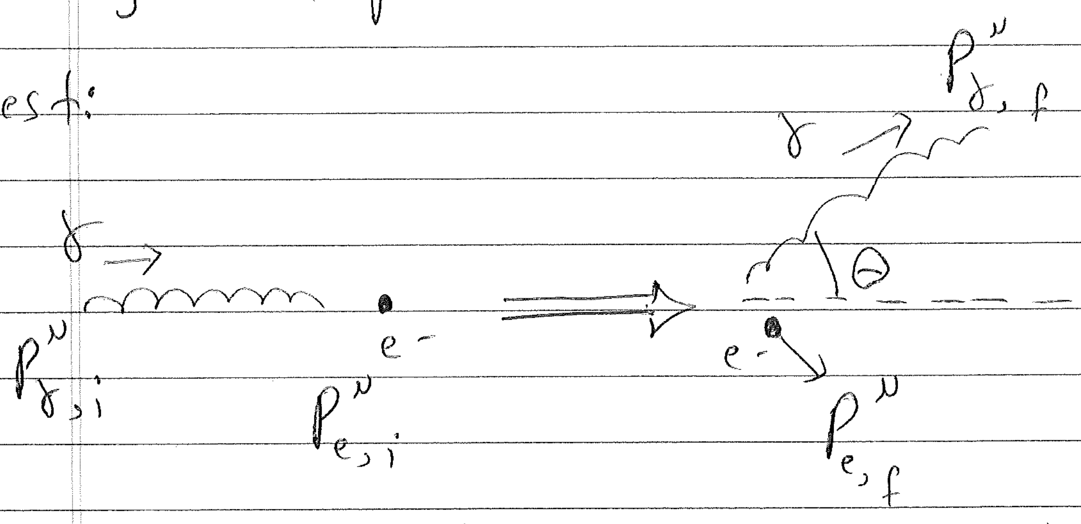
The effect is in marked contrast with Doppler effect in Newtonian dynamics where $\nu' = \nu$ for $\theta = \frac{\pi}{2}$. The transverse Doppler effect in special relativity is the direct consequence of the time-dilation effect.

We also note that in the limit $\frac{v}{c} \ll 1$, the transverse Doppler effect is very small ($\propto \frac{v^2}{c^2}$) as compared with longitudinal Doppler effect when $\theta' \neq \frac{\pi}{2}$ ($\propto \frac{v}{c}$). Since astrophysical objects (like stars) move at a speed $v \ll c$, the transverse Doppler effect will be negligible for them. It is therefore much more difficult to gain information about a binary system through Doppler effect when it is face on.

Kinematics of Compton Scattering:

In special relativity, the total four-momentum of a system is conserved. Conservation of four-momentum combines the separate conservation laws for energy and momentum in Newtonian mechanics, and the relativistic expression for the energy E and momentum $\vec{p}$ ensure that the four-momentum conservation holds in all inertial reference frames.

As an important example, let us consider Compton scattering of a photon off an electron that is initially at rest:



Conservation of the four-momentum of the photon+electron system implies that:

$$p^{\nu}_{\gamma,i} + p^{\nu}_{e,i} = p^{\nu}_{\gamma,f} + p^{\nu}_{e,f} \Rightarrow p^{\nu}_{e,f} = p^{\nu}_{\gamma,i} + p^{\nu}_{e,i} - p^{\nu}_{\gamma,f} \Rightarrow$$

$$p^{\nu}_{e,f} = (p^{\nu}_{\gamma,i} - p^{\nu}_{\gamma,f}) + p^{\nu}_{e,i}$$

We note that:

$$p^{\nu}_{e,i} = (m_e c, 0, 0, 0)$$

$$E_{\gamma,i} = c|\vec{p}_{\gamma,i}| \quad , \quad E_{\gamma,f} = c|\vec{p}_{\gamma,f}|$$

The four-momentum conservation leads to:

$$p^2_{e,f} = \left[(p_{\gamma,i} - p_{\gamma,f}) + p_{e,i}\right]^2$$

Here "2" means contraction of a four-vector with itself, i.e., that $p^2 \equiv (p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2$.

Thus:

$$p^2_{e,f} = \left(\frac{E_{e,f}}{c}\right)^2 - |\vec{p}_{e,f}|^2 = m_e^2 c^2$$

$$\left[(p_{\gamma,i} - p_{\gamma,f}) + p_{e,i}\right]^2 = p^2_{e,i} + (p_{\gamma,i} - p_{\gamma,f})^2 + 2\left(\frac{E_{\gamma,i} - E_{\gamma,f}}{c}\right)\frac{E_{e,i}}{c} - 2(\vec{p}_{\gamma,i} - \vec{p}_{\gamma,f}) \cdot \cancelto{0}{\vec{p}_{e,i}}$$

$$(p_{\gamma,i} - p_{\gamma,f})^2 = \frac{1}{c^2}(E_{\gamma,i} - E_{\gamma,f})^2 - |\vec{p}_{\gamma,i} - \vec{p}_{\gamma,f}|^2$$

Thus:

$$\left[(p_{\gamma,i} - p_{\gamma,f}) + p_{e,i}\right]^2 = p^2_{e,i} + \frac{1}{c^2}(E_{\gamma,i} - E_{\gamma,f})^2 - (|\vec{p}_{\gamma,i}|^2 + |\vec{p}_{\gamma,f}|^2$$

$$2\vec{P}_{\gamma,i}\cdot\vec{P}_{\gamma,f}\Big)+\frac{2}{c^2}\left(E_{\gamma,i}-E_{\gamma,f}\right)E_{e,i}$$

$$P_{e,i}^2=m_e^2c^2\ ,\quad E_{e,i}=m_ec^2$$

since $\vec{P}_{e,i}=0$

Since $|\vec{P}_\gamma|=\frac{E_\gamma}{c}$, and after rearranging the terms in the above expressions, we find:

$$m_ec^2\left(E_{\gamma,i}-E_{\gamma,f}\right)=E_{\gamma,i}\,E_{\gamma,f}\,(1-\cos\theta)$$

$$E_{\gamma,i}=h\nu\ ,\quad E_{\gamma,f}=h\nu'$$

$$\nu'\nu(1-\cos\theta)=hm_ec^2(\nu-\nu')\Rightarrow\nu'=\frac{\nu}{1+\frac{h\nu}{m_ec^2}(1-\cos\theta)}$$

This result implies that the frequency of the scattered photon is always less than or equal to that of the incident photon (in the rest frame of the electron). For forward scattering ($\theta=0$), we have $\nu'=\nu$, while for all other scattering angles $\nu'<\nu$. In the non-relativistic limit $E_{\gamma,i}\ll m_ec^2$, we find $\nu'\simeq\nu$. This is the

Thomson scattering limit.

Gravitational Redshift of Photons:

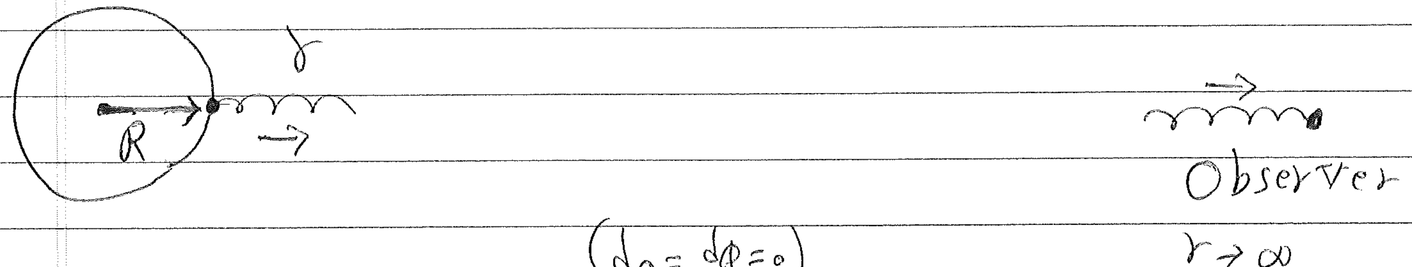
The frequency of a photon changes as it travels through gravitational potential of a distribution of mass and/or energy. This is a general relativistic effect. Here we derive the expression for the gravitational redshift of photons emitted from the surface of a star.

Consider a compact object of mass M and radius R where the mass distribution is spherically symmetric. The spacetime surrounding the mass (i.e., $r > R$) is described by the Schwarzschild metric:

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left[\frac{1}{1 - \frac{2GM}{c^2 r}} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2\right]$$

Here $R_s \equiv \frac{2GM}{c^2}$ is the Schwarzschild radius associated with the mass M. A black hole is formed if $R < R_s$ whose horizon will be at $r = R_s$.

The trajectory of a photon is described by $ds^2 = 0$ (called a "null geodesic"). Now consider two photons that are emitted from the same point on the surface of the star and detected by an observer at infinity:



Considering a radial trajectory ($d\theta = d\phi = 0$), we find:

$$ds^2 = 0 \Rightarrow \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 = \frac{dr^2}{1 - \frac{2GM}{c^2 r}} \Rightarrow c\,dt = \frac{dr}{1 - \frac{2GM}{c^2 r}}$$

$$\Rightarrow \int c\,dt = \int \frac{dr}{1 - \frac{2GM}{c^2 r}}$$

Assuming the first photon is emitted at $t = t_1$ (from $r = R$) and detected at $t = t_1'$ (at $r = R'$), we have:

$$\int_{t_1}^{t'_1} c\,dt = \int_{R}^{R'} \frac{dr}{1-\frac{2GM}{c^2 r}}$$

If the second photon is emitted at $t=t_2$ and detected at $t=t'_2$, we have;

$$\int_{t_2}^{t'_2} c\,dt = \int_{R}^{R'} \frac{dr}{1-\frac{2GM}{c^2 r}}$$

Since the right-hand sides of the two equations in above are equal, we find;

$$\int_{t_1}^{t'_1} c\,dt = \int_{t_2}^{t'_2} c\,dt \Rightarrow t'_2 - t'_1 = t_2 - t_1$$

Calling $\delta t \equiv t_2 - t_1$ and $\delta t' \equiv t'_2 - t'_1$, the emission frequency and the detection frequency, ν and ν' respectively, follow:

$$\nu = \frac{1}{\left(1-\frac{2GM}{c^2 R}\right)^{1/2}\delta t}, \qquad \nu' = \frac{1}{\left(1-\frac{2GM}{c^2 R'}\right)^{1/2}\delta t'}$$

In the limit of weak gravitational field $\frac{GM}{r} \ll c^2$, and hence;

$$\left(1-\frac{2GM}{c^2 r}\right)^{\frac{1}{2}} \simeq 1-\frac{GM}{c^2 r} = 1+\frac{U(r)}{c^2}$$

Here $U(r)$ is the gravitiational potential energy from mass M at a radial distance r. This results in;

$$\frac{\nu'}{\nu} \simeq \frac{1+U(R)}{1+U(R')}$$

As $R' \to \infty$, we have $U(R') \to 0$. Then:

$$\frac{\nu'}{\nu} \simeq 1+U(R) \Rightarrow \frac{\Delta\nu}{\nu} \simeq \frac{GM}{Rc^2} \qquad (\Delta\nu \equiv \nu - \nu')$$

The rehshift factor is defined as:

$$z \equiv \frac{\nu}{\nu'} - 1 \Rightarrow z \simeq \frac{GM}{Rc^2}$$

For photons emitted from the surface of the Sun ($M = M_\odot$, $R = R_\odot \sim 10^6$ km), we find;

$$z_{Sun} \sim O(10^{-6})$$

For a white dwarf ($M_{WD} \sim M_\odot$, $R_{WD} \sim 10^4$ km), we find:

$$z_{WD} \sim O(10^{-4})$$

For a neutron star ($M_{NS} \sim M_\odot$, $R_{NS} \sim 10$ km), we find;

$z_{NS} \sim O(10^{-1})$

The redshift is an easily measurable quantity in this case. For example, in the low mass X-ray binary system EXO 0748-676, X-ray burst spectra often contain strong absorption lines identified with Iron and Oxygen transitions. All of the lines are redshifted by a fractional amount ~ 0.35, which has been interpreted as a gravitational redshift on the surface of a $\sim 2M_\odot$ neutron star.